ON A CLASS OF MOTIONS OF CONSERVATIVE SYSTEMS WITH ONE NONCYCLIC COORDINATE

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PMM Vol.30, № 4, 1966, pp.617-624

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(Received January 18, 1966)

Motions of conservative systems which, in a certain sense, generalize the precessional and nutational motions of gyroscopes, are investigated. Fairly simple conditions of stability of precessional motions in terms of all velocities and of noncyclic coordinate, are obtained. These conditions are, as a rule, necessary and sufficient. In one particular case, they are reduced to well known conditions of stability for the gyroscopes with a vertical external suspension. The instability of the system is estimated in terms of cyclic coordinates and of the vector of mean deviation of the system from nonperturbed precession during the period of one nutational oscillation (Magnus type time drift).

1. Let a conservative system with $n+1$ degrees of freedom $(n \geq 1)$ possess generalized coordinates $\alpha_1, \ldots, \alpha_n, \beta$, where β is the noncyclic coordinate, while the remaining ones are cyclic. Then, the kinetic potential will be

$$L(\alpha_1^{\cdot}, \ldots, \alpha_n^{\cdot}, \beta^{\cdot}, \beta) = \frac{1}{2} \sum_{i,\, j=1}^{n} a_{ij}(\beta)\alpha_i^{\cdot}\alpha_j^{\cdot} + \beta^{\cdot} \sum_{i=1}^{n} a_i(\beta)\alpha_i^{\cdot} + \frac{1}{2} b(\beta)\beta^{\cdot 2} - \Pi(\beta)$$

where $\Pi(\beta)$ is the potential energy of the system. Symmetric matrices

$$A(\beta) = \begin{Vmatrix} a_{11}, & \ldots, & a_{1n}, & a_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}, & \ldots, & a_{nn}, & a_n \\ a_1, & \ldots, & a_n, & b \end{Vmatrix}, \qquad B(\beta) = \begin{Vmatrix} a_{11}, & \ldots, & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1}, & \ldots, & a_{nn} \end{Vmatrix}$$

are positive-definite (in the sense of the corresponding quadratic forms) over some interval of variation of β. We assume that the functions $a_{ij}(\beta)$, $a_i(\beta)$, $b(\beta)$, $\Pi(\beta)$ are continuous and possess derivatives of any order.

Let $C(\beta)$ be the inverse of B, i.e. $C = B^{-1}$ and let us introduce n-dimensional vectors $a = (a_1, \ldots, a_n)$, $\alpha = (\alpha_1^{\cdot}, \ldots, \alpha_n^{\cdot})$. Then, the kinetic potential becomes

$$L(\alpha^{\cdot}, \beta^{\cdot}, \beta) = {}^1/_2\,(B\alpha^{\cdot}, \alpha^{\cdot}) + \beta^{\cdot}(a, \alpha^{\cdot}) + {}^1/_2\, b\beta^{\cdot 2} - \Pi(\beta)$$

where (x, y) is the scalar product of the vectors x and y.

Let us now consider the motion with initial conditions β_0, $\beta_0^{\cdot}$, $\alpha_0^{\cdot}$; in the following, A_0, B_0, ..., a_0 and b_0 will denote values of the functions of β, when $\beta = \beta_0$.

Lagrange's equations yield n first integrals

$$B\alpha^{\cdot} + \beta^{\cdot} a = \Delta = B_0\alpha_0^{\cdot} + \beta_0^{\cdot} a_0 \qquad (1.1)$$

and the energy integral

$$\begin{aligned} {}^1/_2(B\alpha^{\cdot}, \alpha^{\cdot}) + \beta^{\cdot}(a, \alpha^{\cdot}) + {}^1/_2 b\beta^{\cdot 2} + \Pi(\beta) = {}^1/_2\,\varepsilon = \\ = {}^1/_2(B_0\alpha_0^{\cdot}, \alpha_0^{\cdot}) + \beta_0^{\cdot}(a_0, \alpha_0^{\cdot}) + {}^1/_2 b_0\beta_0^{\cdot 2} + \Pi_0 \end{aligned} \qquad (1.2)$$

From (1.1) we have

$$\alpha^{\cdot} = B^{-1}(\Delta - \beta^{\cdot} a) = C\Delta - \beta^{\cdot} Ca \qquad (1.3)$$

and the energy integral becomes

$$\begin{aligned} {}^1/_2(\Delta - \beta^{\cdot} a, C\Delta - \beta^{\cdot} Ca) + \beta^{\cdot}(a, C\Delta - \beta^{\cdot} Ca) + {}^1/_2 b\beta^{\cdot 2} + \Pi(\beta) = \\ = {}^1/_2[b - (Ca, a)]\beta^{\cdot 2} + {}^1/_2(C\Delta, \Delta) + \Pi(\beta) = {}^1/_2\,\varepsilon \end{aligned}$$

Let $|A|$ and $|B|$ be the determinants of the matrices A and B. Since $|A| > 0$ and $|B| > 0$, it is easy to see that

$$b - (Ca, a) = \frac{|A|}{|B|} > 0$$

After introducing the notations

$$h(\beta) \equiv \frac{|A|}{|B|}, \qquad \varphi(\beta, \Delta) = (C\Delta, \Delta) + 2\Pi(\beta)$$

the energy integral assumes the form

$$h(\beta)\beta^{\cdot 2} + \varphi(\beta, \Delta) = \varepsilon = h_0\beta_0^{\cdot 2} + \varphi(\beta_0, \Delta) \qquad (1.4)$$

In the following, differentiation with respect to β will be denoted by a prime; thus $h'(\beta)$, $\varphi'(\beta, \Delta)$, C' etc. Then, the Routh equation of motion will be

$$2h(\beta)\beta^{\cdot\cdot} + h'(\beta)\beta^{\cdot 2} + \varphi'(\beta, \Delta) = 0$$

If the initial values of β_0, $\beta_0^{\cdot} = 0$ and $\alpha_0^{\cdot}$ are such that

$$\varphi'(\beta_0, \Delta) = 0, \quad \text{for} \quad (C_0'\Delta, \Delta) + 2\Pi_0' = 0 \qquad (1.5)$$

then the motion of the type $\beta \equiv \beta_0$, $\alpha^{\cdot} = \text{const}$ exists, and we shall call it, in conformity with gyroscopic terminology, the precession. For the precession we have, from (1.3), $\alpha^{\cdot} = \alpha_0^{\cdot} = C_0\Delta$.

2. Let us consider a line $z = \varphi(\beta, \Delta)$ and a horizontal straight line $z = \epsilon$ where Δ and ϵ are defined in terms of some arbitrary initial conditions β_0, $\beta_0^{\cdot}$ and α_0, on the plane with rectangular coordinates (β, z) where β is the horizontal and z is the vertical axis. By (1.4), we always have $\varphi(\beta_0, \Delta) \leq \epsilon$. $\varphi(\beta_0, \Delta) = \epsilon$ if and only if $\beta_0^{\cdot} = 0$ (since $h_0 > 0$). If $\beta_0^{\cdot} \neq 0$, then on some interval containing β_0, the inequality $\varphi(\beta, \Delta) < \epsilon$ holds. Now let us assume, that between the lines $z = \epsilon$ and $z = \varphi(\beta, \Delta)$, points β_1 and β_2 exist such, that $\varphi(\beta_1, \Delta) = \epsilon$, $\varphi(\beta_2, \Delta) = \epsilon$ and $\varphi(\beta, \Delta) < \epsilon$ for $\beta_1 < \beta < \beta_2$. It is easily seen that, if the line

$z = \varepsilon$ intersects the curve $z = \varphi(\beta, \Delta)$ at two distinct points β_1 and β_2 (i.e. $\varphi'(\beta_1, \Delta) < 0$ and $\varphi'(\beta_2, \Delta) > 0$), then the Routh equation has a periodic solution, and its period

$$T = 2\int_{\beta_1}^{\beta_2} \frac{\sqrt{h(\beta)}\, d\beta}{\sqrt{\varepsilon - \varphi(\beta, \Delta)}} \tag{2.1}$$

assumes all possible values between β_1 and β_2. By (1.3) is also a periodic function of time t, and has the same period T. We shall call the above motion nutational.

In case of nutational motion, function $\varphi(\beta, \Delta)$ has at least one minimum in the interval $\beta_1 < \beta < \beta_2$. The nutation will be called proper nutation if $\varphi(\beta, \Delta)$ has exactly one minimum. In this case the minimum point will be called the center of nutation, and denoted by β_+. Obviously, $\varphi'(\beta_+, \Delta) = 0$, hence $\beta \equiv \beta_+$, while $\dot{\alpha} \equiv C(\beta_+)\Delta$ will be a precession corresponding to the center of nutation.

Let us now represent (1.3) and (1.4), as

$$\begin{aligned} \alpha^{\cdot} &= [C(\beta) - C(\beta_+)]\,\Delta + C(\beta_+)\,\Delta - \beta^{\cdot} Ca \\ h(\beta)\,\beta^{\cdot 2} + [\varphi(\beta, \Delta) - \varphi(\beta_+, \Delta)] &= \nu^2 = \varepsilon - \varphi(\beta_+, \Delta) \end{aligned} \tag{2.2}$$

We should note that the limits β_1 and β_2 of variation of β are roots of Equation

$$\varphi(\beta, \Delta) - \varphi(\beta_+, \Delta) = \nu^2 \tag{2.3}$$

while the period T is given by (2.1), in which $\varepsilon - \varphi(\beta, \Delta)$ is replaced by $\nu^2 - [\varphi(\beta, \Delta) - \varphi(\beta_+, \Delta)]$. Let us now determine the increment of the vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ during one period of nutation. Integrating (2.2) with respect to t and replacing the variable t by β, we find

$$\delta\alpha = 2\int_{\beta_1}^{\beta_2} \frac{\sqrt{h(\beta)}\,[C(\beta) - C(\beta_+)]\,\Delta}{\sqrt{\nu^2 - [\varphi(\beta, \Delta) - \varphi(\beta_+, \Delta)]}}\, d\beta + TC(\beta_+)\,\Delta$$

since the integral of $\beta^{\cdot} Ca$ over one complete period, vanishes.

Mean increment of vector α per one period

$$\frac{\delta\alpha}{T} \equiv \langle\alpha^{\cdot}\rangle = \frac{2}{T}\int_{\beta_1}^{\beta_2} \frac{\sqrt{h(\beta)}\,[C(\beta) - C(\beta_+)]\,\Delta}{\sqrt{\nu^2 - [\varphi(\beta, \Delta) - \varphi(\beta_+, \Delta)]}}\, d\beta + C(\beta_+)\,\Delta \tag{2.4}$$

The center β_+ of nutation is a root of (1.5) and depends only on the choice of Δ. Integral in (2.4) depends on Δ and ν^2. Let us denote it by $\Phi(\nu^2, \Delta)$. Then we can represent (2.4) as

$$\langle\alpha^{\cdot}\rangle = \Phi(\nu^2, \Delta) + C(\beta_+(\Delta))\,\Delta \tag{2.5}$$

An approximate expression for $\Phi(\nu^2, \Delta)$ for small ν and fixed Δ will be given later, while now we shall consider the stability of precession.

3. From (1.5) it follows that, for the precession $\beta = \beta_0$, $\alpha^{\cdot} = C_0\Delta$, the curve $z = \varphi(\beta, \Delta)$ has a horizontal tangent at the point $\beta = \beta_0$. If, at the same time

$$\varphi''(\beta_0, \Delta) > 0, \quad \text{for} \quad (C_0''\Delta, \Delta) + 2\Pi_0'' > 0 \tag{3.1}$$

is fulfilled (i.e. if β_0 is the minimum point of $\varphi(\beta, \Delta)$), then the precession is stable in β and $\alpha^\cdot$. This easily follows from the assumption of continuity of a_{ij}, a_i, b and Π and of their derivatives as functions of β, and from the conditions (1.5) and (3.1). If $\varphi''(\beta, \Delta) < 0$, then the precession is unstable. The case $\varphi''(\beta_0, \Delta) = 0$ shall not be considered.

Let us now assume that β_0 and Δ satisfy (1.5) and (3.1). Then, the corresponding precession is stable in β and $\alpha^\cdot$, although as a rule, it is not stable in α. The latter follows from (2.5). Below we shall show, that $\Phi(\nu^2, \Delta)$ differs, as a rule, from the null-vector, hence some of its terms are different from zero. This means that the value of the corresponding coordinate α_i departs systematically from its precessional value. Later we shall obtain a quantitative estimate of this instability for various types of perturbations, while now we shall give some examples of application of Formulas (1.5) and (3.1).

4. Examples. Let us consider a gyroscope on gimbals with a vertical axis of the outer frame. We shall also consider gyroscopes with intersecting or crossing frame axes, where the center of gravity of the gyroscope is displaced downward along the axis of the rotor. We shall denote by α_1 the angle of rotation of the outer frame, α_2 will be the angle of rotation of the rotor and β will be the angle of rotation of the inner frame. We have $\Pi(\beta) = mgl \sin\beta$ where m is the mass of the rotor with the inner frame, and l is the displacement of the combined center of gravity of the rotor and the inner frame, along the axis of the rotor. Kinetic energy of the gyroscope is given by

$$2T = M_1(\beta)\alpha_1^{\cdot 2} + J(\alpha_2^\cdot + \alpha_1^\cdot \sin\beta)^2 - 2\alpha_1^\cdot\beta^\cdot N_1(\beta) + I\beta^{\cdot 2}$$

$$M_1(\beta) = p_1 + q_1\cos 2\beta - r_1 \sin 2\beta - s_1\cos\beta, \qquad N_1(\beta) = D_1\cos\beta + K_1\sin\beta$$

Here p_1, q_1, r_1, s_1 D_1, K_1 and I are various moments of inertia (axial and centrifugal) of the frames, rotor and their linear combinations, while J is the axial moment of the rotor. For heavy gyroscope on gimbals $r_1 = s_1 = 0$, for a gyroscope with intersecting axes $s_1 = 0$, while for the gyroscope in which the combined center of gravity of the rotor and the inner frame is not displaced with respect to the axis of the rotation of the inner ring, we have $l = 0$.

First integrals of motion have the form

$$\Delta_1 = M_1(\beta)\alpha_1^\cdot + J(\alpha_2^\cdot + \alpha_1^\cdot\sin\beta)\sin\beta - N_1(\beta)\beta^\cdot$$

Hence

$$\Delta_2 = J(\alpha_2^\cdot + \alpha_1^\cdot\sin\beta) \equiv J\Omega \qquad (M_1(\beta) > 0)$$

$$a_{11} = M_1(\beta) + J\sin^2\beta, \quad a_{12} = a_{21} = J\sin\beta, \quad a_{22} = J, \quad a_1 = -N_1(\beta), \quad a_2 = 0, \quad b = I$$

We assume that $M_1(\beta) > 0$ for all values of β. We easily obtain

$$\varphi(\beta, \Delta) = (C\Delta, \Delta) + 2\Pi(\beta) = \frac{(\Delta_1 - \Delta_2\sin\beta)^2}{M_1(\beta)} + \frac{\Delta_2^2}{J} + 2mgl\sin\beta$$

Disregarding the uninteresting case $\Omega = 0$, we shall assume $\Omega \neq 0$. Let us introduce the dimensionless parameters

$$\mu = \frac{2mgl}{J\Omega^2}, \quad \omega = \frac{\alpha_1^\cdot}{\Omega}, \quad \Delta = \frac{\Delta_1}{\Delta_2}, \quad p = \frac{p_1}{J}, \quad q = \frac{q_1}{J}, \quad r = \frac{r_1}{J}, \quad s = \frac{s_1}{J}$$

$$D = \frac{D_1}{J}, \quad K = \frac{K_1}{J}$$

and functions

$$M(\beta) = \frac{M_1(\beta)}{J} = p + q\cos 2\beta - r\sin 2\beta - s\cos\beta$$

$$N(\beta) = \frac{N_1(\beta)}{J} = D\cos\beta + K\sin\beta$$

Then, we shall have

$$\varphi(\beta, \Delta) = J\Omega^2\left[1 + \frac{(\sin\beta - \Delta)^2}{M(\beta)} + \mu\sin\beta\right]$$

In the following, instead of the minimum of $\varphi(\beta, \Delta)$, we shall try to obtain the minimum of

$$\psi(\beta, \Delta) = \frac{(\sin\beta - \Delta)^2}{M(\beta)} + \mu\sin\beta \qquad (4.1)$$

where

$$\Delta = \frac{\Delta_1}{\Delta_2} = \frac{M_1(\beta)\alpha_1{}^{\cdot} + J\Omega\sin\beta - N_1(\beta)\beta^{\cdot}}{J\Omega} = \sin\beta + \omega M(\beta) - N(\beta)\frac{\beta^{\cdot}}{\Omega}$$

Consequently, if the initial condition is such that $\beta_0{}^{\cdot} = 0$, we have

$$\Delta = \sin\beta_0 + M_0\omega_0 \qquad (4.2)$$

Differentiating (4.1), we obtain

$$\psi'(\beta, \Delta) = \frac{2(\sin\beta - \Delta)\cos\beta}{M(\beta)} - \frac{(\sin\beta - \Delta)^2 M'(\beta)}{M^2(\beta)} + \mu\cos\beta$$

Replacing β with β_0, taking (4.2) into account and equating the resulting expression to zero, we obtain the equation for ω_0, the angular velocity of precession of the outer frame. We have

$$(\mu - 2\omega_0)\cos\beta_0 - M_0'\omega_0^2 = 0 \qquad (4.3)$$

This quadratic equation has real roots, if

$$\cos^2\beta_0 + \mu M_0'\cos\beta_0 \geqslant 0 \qquad (4.4)$$

For a heavy gyroscope ($r = s = 0$), we have

$$M_0' = -4q\sin\beta_0\cos\beta_0$$

hence (4.3) is satisfied at any value of ω_0, if $\beta_0 = \pm\frac{1}{2}\pi$ (existence of arbitrary precessions of a heavy gyroscope with folded frames). We have

$$\psi''(\beta, \Delta) = \frac{2\cos^2\beta}{M(\beta)} - \frac{2(\sin\beta - \Delta)\sin\beta}{M(\beta)} - \frac{4(\sin\beta - \Delta)\cos\beta M'(\beta)}{M^2(\beta)} -$$
$$- \frac{(\sin\beta - \Delta)^2 M''(\beta)}{M^2(\beta)} + 2\frac{(\sin\beta - \Delta)^2[M'(\beta)]^2}{M^3(\beta)} - \mu\sin\beta$$

Assuming that $\beta = \beta_0$ in $\psi''(\beta, \Delta)$, utilizing the relation (4.2) and replacing according to (4.3) $2\omega_0\cos\beta_0$ by $\mu\cos\beta_0 - M_0'\omega_0^2$, we obtain the following condition of stability

$$2M_0^{-1}(\cos^2\beta_0 + \mu M_0'\cos\beta_0) + [(2\omega_0 - \mu)\sin\beta_0 - M_0''\omega_0^2] > 0 \qquad (4.5)$$

while the condition that the precession $\beta \equiv \beta_0$, $\omega \equiv \omega_0$, is unstable, is

$$2M_0^{-1}(\cos^2\beta_0 + \mu M_0'\cos\beta_0) + [(2\omega_0 - \mu)\sin\beta_0 - M_0''\omega_0^2] < 0 \qquad (4.6)$$

from which we see that stability is independent of $N(\beta)$.

Let us mention few particular cases.

1) Since $M_0 > 0$, by (4.4) the condition (4.5) is fulfilled if

$$(2\omega_0 - \mu)\sin\beta_0 - M_0''\omega_0^2 > 0$$

This sufficient condition of stability was obtained by Sinitsin [1].

2) For a heavy gyroscope we have

$$M_0 = p + q\cos 2\beta_0 = (p + q) - 2q\sin^2\beta_0, \qquad M_0' = -4q\sin\beta_0\cos\beta_0$$

$$M_0'' = -4q\cos 2\beta_0$$

Assuming $\beta_0 \neq \pm\frac{1}{2}\pi$ we find from (4.3) that $\mu = 2\omega_0 - 4q\sin\beta_0\omega_0^2$. Substituting this into (4.5), multiplying the result by M_0 and dividing it by $2\cos^2\beta_0$, we shall obtain the condition of stability of precession of a heavy gyroscope which was previously found by Skimel' [2]. In our notation it has the form

$$1 - 8q\omega_0 \sin\beta_0 + 2q\omega_0^2 (p + q + 6q \sin^2\beta_0) > 0$$

For $\beta_0 = \pm \frac{1}{2}\pi$ we have

$$M_0 = p - q, \quad M_0' = -4q \cos\beta_0 \sin\beta_0 = 0, \quad M_0'' = 4q$$

Here ω_0 is arbitrary. The condition of stability $\pm(2\omega_0 - \mu) - 4q\omega_0^2 > 0$ (+ sign for $\beta = +\frac{1}{2}\pi$, − sign for $\beta_0 = -\frac{1}{2}\pi$) was found previously by Rumiantsev [3] and Magnus [4].

3) If $2r \pm s \neq 0$, then $M'(\pm \frac{1}{2}\pi) \neq 0$. Assuming the last condition fulfilled we find from (4.3) that, when $\beta_0 = \pm\frac{1}{2}\pi$, then $\omega_0 = 0$. Hence the precession $\beta_0 = \pm\frac{1}{2}\pi$, $\omega_0 = 0$ is stable for $\mp\mu > 0$, and unstable for $\pm\mu < 0$.

4) Let β_0 be such, that $M_0' = 0$. Then $\omega_0 = \frac{1}{2}\mu$ and the precession $\beta \equiv \beta_0$, $\omega = \frac{1}{2}\mu$ is stable if $M_0'' < 0$ (assuming that $\mu \neq 0$), i.e. if β_0 is the maximum of $M(\beta)$.

5) Let $\mu = 0$ (gyroblock is in static equilibrium with respect to its axis of rotation). Then, one of the roots of (4.3) is equal to zero: $\omega_0 = 0$. From (4.5) we find that the precession $\beta \equiv \beta_0$, $\omega_0 = 0$ is stable for $\beta_0 \neq \mp\frac{1}{2}\pi$.

In the above samples we have considered the stability with respect to angular velocities of the rotor, of the outer and inner frame and with respect to the angle of rotation of the inner frame. We know that even the precession of the example (5) is unstable with respect to the angle of rotation of the outer frame. (Magnus type time drift). Let us now return to the quantitative estimate of the mean drift of a_1 per one period of nutational oscillation, defined by (2.5).

5. Let β_0 and Δ be such, that the conditions (1.5) and (3.1) are satisfied. Then, for sufficiently small $|\beta_0^{\cdot}|$, the initial conditions

$$\beta = \beta_0, \qquad \alpha^{\cdot} = C_0(\Delta - \beta_0^{\cdot} a_0), \qquad \beta^{\cdot} = \beta_0^{\cdot}$$

result in a proper nutation with its center at β_0. Let us find the first term of the expansion of $\Phi(\nu^2, \Delta)$ in powers of ν^2. To do this, we shall introduce in the integrals (2.1) and (2.4) a new variable of integration u, assuming that

$$\beta = \beta(u) \equiv {}^1/_2(\beta_2 + \beta_1) + {}^1/_2(\beta_2 - \beta_1)\sin u$$

where β_1 and β_2 are roots of (2.3).

In the following we shall denote by $O(\nu)$ and $O(\nu, u)$ functions (vectors or matrices the components of which are functions) defined and uniformly bounded on the interval $|\nu| \leq \delta$ or in the rectangle $|\nu| \leq \delta$, $|u| \leq \frac{1}{2}\pi$, respectively, where $\delta > 0$ is a sufficiently small number.

Let us represent (2.3) in the form

$${}^1/_2\,\varphi_0''(\beta - \beta_0)^2 + {}^1/_6\varphi_0'''(\beta - \beta_0)^3 + k(\beta)(\beta - \beta_0)^4 = \nu^2$$

where $k(\beta)$ is a function bounded in the vicinity of the point $\beta = \beta_0$. Then, we easily find

$$\begin{aligned}\beta_1 &= \beta_0 - c|\nu| + d\nu^2 + \nu^3 O(\nu)\\ \beta_2 &= \beta_0 + c|\nu| + d\nu^2 + \nu^3 O(\nu)\end{aligned} \qquad \left(c = \left(\frac{2}{\varphi_0''}\right)^{1/2}, \quad d = -\frac{r\,\varphi_0'''}{3(\varphi_0'')^2}\right) \tag{5.1}$$

and

$$\beta(u) = \beta_0 + c|\nu|\sin u + d\nu^2 + \nu^3 O(\nu, u) \tag{5.2}$$

Consequently, we have for an arbitrary function $f(\beta)$

$$f(\beta(u)) = f_0 + f_0'c|\nu|\sin u + (df_0' + {}^1/_2\, c^2 f_0'' \sin^2 u)\nu^2 + \nu^3 O(\nu, u) \quad (5.3)$$

By (1.5) and (2.3) we can show, that for the function $\varphi(\beta, \Delta)$,

$$\nu^2 - [\varphi(\beta, \Delta) - \varphi(\beta_0, \Delta)] = \left[1 + \frac{c\varphi_0'''}{3\varphi_0''}|\nu|\sin u + \nu^2 O(\nu, u)\right]\nu^2\cos^2 u$$

from which

$$\frac{|\nu|\cos u}{\sqrt{\nu^2 - [\varphi(\beta, \Delta) - \varphi(\beta_0, \Delta]}} = 1 - \frac{c\varphi_0'''}{6\varphi_0''}|\nu|\sin u + \nu^2 O(\nu; u) \quad (5.4)$$

follows.

After the change of variable in (2.1) and (2.4), new integrand functions will, utilizing (5.3) for $\sqrt{h(\beta)}$ and $(C\beta)$ and (5.4) for $\varphi(\beta, \Delta)$, be in the integral (2.1)

$$\frac{\beta_2 - \beta_1}{2|\nu|}\,\frac{|\nu|\cos u\sqrt{h}}{\sqrt{\nu^2 - \varphi + \varphi_0}} =$$

$$= (c + \nu^2 O(\nu))\left[\sqrt{h_0} + |\nu|c\left(\frac{h_0'}{2\sqrt{h_0}} - \frac{\varphi_0'''}{6\varphi_0''}\right)\sin u + \nu^2 O(\nu, u)\right]$$

and in the integral (2.4)

$$\frac{\beta_2 - \beta_1}{2|\nu|}\,\frac{|\nu|\cos u}{\sqrt{\nu^2 - \varphi + \varphi_0}}\sqrt{h}(C - C_0) = (c + \nu^2 O(\nu))\Big\{|\nu|\sqrt{h_0}\,C_0'c\sin u +$$

$$+ \nu^2\left[c^2 C_0'\left(\frac{'h_0'}{2\sqrt{h_0}} - \frac{\varphi_0'''}{6\varphi_0''}\right)\sin^2 u + \right.$$

$$\left. + \sqrt{h_0}\left(dC_0' + \frac{c^2}{2}C_0''\sin^2 u\right)\right] + \nu^3 O(\nu, u)\Big\}$$

Since the integral of $\sin u$, $\sin^2 u$ and 1 is, over the interval $-\frac{1}{2}\pi \le u \le \frac{1}{2}\pi$, equal to 0 , $\frac{1}{2}\pi$ and π , respectively, we can easily obtain, utilizing (5.1),

$$T = 2\pi\sqrt{\frac{2h_0}{\varphi_0''}} + \nu^2 O(\dot{\nu})$$

$$T\Phi(\nu^2, \Delta) = 2\pi\left(\frac{2}{h_0\varphi_0''}\right)^{1/2}\left[h_0\left(\frac{C_0''}{2\varphi_0''} - \frac{\varphi_0'''C_0'}{3(\varphi_0'')^2}\right) + \frac{C_0'}{2\varphi_0''}\left(h_0' - \frac{\varphi_0'''h_0}{3\varphi_0''}\right)\right]\nu^2\Delta +$$

$$+ \nu^3 O(\nu) = \pi\left(\frac{2}{h_0\varphi_0''}\right)^{1/2}\left(\frac{hC'}{\varphi''}\right)_0'\Delta\nu^2 + \nu^3 O(\nu) \quad (5.5)$$

It can also be easily shown, that the last term can be replaced by $\nu^4 O(\nu)$. Hence, we have

$$\Phi(\nu^2, \Delta) = \frac{\nu^2}{2h_0}\left(\frac{hC'}{\varphi''}\right)_0'\Delta + \nu^4 O(\nu) = \frac{\beta_0^{\cdot 2}}{2}\left(\frac{hC'}{\varphi''}\right)_0'\Delta + \beta_0^{\cdot 4}O(\beta_0^{\cdot})$$

Here $\beta_0^{\cdot}$ is the rate of nutation at the instant of crossing the center of nutation. By (2.5), we have

$$\langle\alpha^{\cdot}\rangle = C_0\Delta + \frac{1}{2}\beta_0^{\cdot 2}\left(\frac{hC'}{\varphi''}\right)_0'\Delta + \beta_0^{\cdot 4}O(\beta_0^{\cdot}) \quad (5.6)$$

Let us use (5.6) to find the drift of the external frame of the gyroscope in static equilibrium with respect to its axis ($\ell = \mu = 0$) . In this case we have the precession $\beta \equiv \beta_0$, $w_0 = 0$. Using the notation of Section 4 we find, by (5.6),

$$\langle\alpha_1^{\boldsymbol{\cdot}}\rangle =$$

$$= -\beta_0{}^{\boldsymbol{\cdot}}{}_2 \frac{I[2(p-q)\sin\beta_0 - (2r+s\sin\beta_0)\cos\beta_0] - JN_0(K+N_0\sin\beta_0)}{4J\Omega M_0\cos^2\beta_0} + \beta^{\boldsymbol{\cdot}}{}_0{}^4 O(\beta_0^{\boldsymbol{\cdot}})$$

When $r = s = D = K = 0$, we have the known Magnus approximate formula for the drift of external frame.

6. Let β_1 and the vector $\Delta^{(1)}$ satisfy the conditions (1.5) and (3.1). Then the stable precession $\beta \equiv \beta_1,\ \alpha^{\boldsymbol{\cdot}} \equiv \alpha^{\boldsymbol{\cdot}(1)} = C(\beta_1)\Delta^{(1)}$ exists. Let β_2 and $\Delta^{(2)}$ be an arbitrary number and vector, respectively, such, that $|\beta_2-\beta_1|$ and $|\Delta^{(2)} - \Delta^{(1)}|$ are sufficiently small, and let $\beta_2^{\boldsymbol{\cdot}}$ be also sufficiently small. Then the motion with initial conditions β_2 , $\Delta^{(2)}$ and $\beta_2^{\boldsymbol{\cdot}}$ will be a proper nutation, the center of which we shall denote by β_0 . The rate $\beta_0^{\boldsymbol{\cdot}}$ of this nutation at the instant of crossing the center of nutation, can be found from

$$h_0\beta_0^{\boldsymbol{\cdot}2} = h(\beta_2)\beta_2^{\boldsymbol{\cdot}2} + \varphi(\beta_2, \Delta^{(2)}) - \varphi(\beta_0, \Delta^{(2)})$$

and it can be shown that $\beta_0^{\boldsymbol{\cdot}2}$ is a second order infinitesimal with respect to $|\beta_2^{\boldsymbol{\cdot}}| + |\beta_2 - \beta_1| + |\Delta^{(2)} - \Delta^{(1)}|$.

Let us now find the vector of mean deviation $\langle\alpha^{\boldsymbol{\cdot}} - \alpha_1^{\boldsymbol{\cdot}}\rangle$. By (5.6) we have

$$\langle\alpha^{\boldsymbol{\cdot}} - \alpha^{\boldsymbol{\cdot}(1)}\rangle = C_0\Delta^{(2)} - C(\beta_1)\Delta^{(1)} + \frac{1}{2}\beta_0^{\boldsymbol{\cdot}2}\left(\frac{hC'}{\varphi''}\right)_0'\Delta^{(2)} + \beta_0^{\boldsymbol{\cdot}4}O(\beta_0^{\boldsymbol{\cdot}}) \quad (6.1)$$

Let us estimate $C_0\Delta^{(2)} - C(\beta_1)\Delta^{(1)}$. Putting $\xi = \Delta^{(2)} - \Delta^{(1)}$, we shall note that

$$\varphi(\beta, \Delta^{(2)}) = (C\Delta^{(2)}, \Delta^{(2)}) + 2\Pi(\beta) = (C(\Delta^{(1)}+\xi), \Delta^{(1)}+\xi) + $$
$$+ 2\Pi(\beta) = \varphi(\beta, \Delta^{(1)}) + 2(C\Delta^{(1)}, \xi) + (C\xi, \xi)$$

Hence, by (1.5) we can find the center of nutation using

$$\varphi'(\beta_0, \Delta^{(1)}) + 2(C_0'\Delta^{(1)}, \xi) + (C_0'\xi, \xi) = 0$$

For small $|\xi|$ this equation has, by virtue of $\varphi'(\beta_1, \Delta^{(1)}) = 0$ a solution

$$\beta_0 = \beta_1 - \frac{2(C'(\beta_1)\Delta^{(1)}, \xi)}{\varphi''(\beta_1, \Delta^{(1)})} + |\xi|^2 O(|\xi|)$$

Hence,

$$C_0\Delta^{(2)} - C(\beta_1)\Delta^{(1)} = (C(\beta_1) + C'(\beta_1)(\beta_1-\beta_0) + |\xi|^2 O(|\xi|))(\Delta^{(1)}+\xi) -$$
$$- C(\beta_1)\Delta^{(1)} = C(\beta_1)\xi + \frac{2(C'(\beta_1)\Delta^{(1)}, \xi)}{\varphi''(\beta_1, \Delta^{(1)})}C'(\beta_1)\Delta^{(1)} + |\xi|^2 O(|\xi|)$$

From this we see, that, if $|\xi|$, $|\beta_2 - \beta_1|$ and $\beta^{\boldsymbol{\cdot}}_2$ are of the same order of magnitude, then $C_0\Delta^{(2)} - C(\beta_1)\Delta^{(1)}$; will be the principal term of (5.7) and we shall have

$$\langle\alpha^{\boldsymbol{\cdot}} - \alpha^{\boldsymbol{\cdot}(1)}\rangle \approx C(\beta_1)\xi + \frac{2(C'(\beta_1)\Delta^{(1)}, \xi)}{\varphi''(\beta_1, \Delta^{(1)})}C'(\beta_1)\Delta^{(1)}$$

If $\xi = 0$, i.e. $\Delta^{(2)} = \Delta^{(1)}$, $\beta_0 = \beta_1$ (center of nutation coincides with the initial point β_1), then the following approximate formula can be used:

$$\langle \alpha^{\cdot} - \alpha^{\cdot(1)} \rangle \approx \frac{\nu^2}{2h(\beta_1)} \left(\frac{hC'}{\varphi''} \right)'_{\beta=\beta_1} \Delta^{(1)}$$

$$\left(\nu^2 \approx h(\beta_2)\, \beta_2^{\cdot 2} + \frac{1}{2} \varphi''(\beta_1, \Delta^{(1)}) (\beta_2 - \beta_1)^2 \right)$$

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Translated by L.K.